Homework 13

Math 622

May 12, 2014

1. a) Write out a forward LIBOR model—see (10.4.19) in Shreve—for $n = 3$ and $\delta = 0.25$, with $\gamma(t, T_1) = 1$, $\gamma(t, T_2) = 3$, and $\gamma(t, T_3) = 2$, on a probability space $(\omega, \mathcal{F}, \widetilde{\mathbf{P}}^{T_4})$ with a given Brownian motion $\widetilde{W}^{T_4}$. Show how to define the T_3-forward measure, $\widetilde{\mathbf{P}}^{T_3}$ by a change of measure and also how to define the Brownian motion $\widetilde{W}^{T_3}$ under $\widetilde{\mathbf{P}}^{T_3}$.

b) Derive an HJM model consistent with this forward LIBOR model. This requires you to define $\sigma^*(t, T_j)$, $j = 1, \ldots, 4$ and to construct, $\widetilde{\mathbf{P}}$ and $\widetilde{W}$, the risk-neutral measure for prices in dollars and the associate Brownian motion; see Theorem 10.4.4 in Shreve. You have freedom in the definition of $\sigma^*(t, T)$. Choose it to be continuous in t.

Discussion (Read this before doing problems 2 and 3.) Exercise 10.11, which you did in a previous assignment, develops a basic formula for swaps. In the swap business, floating rate payments are referred to as *floating legs* and fixed rate payments as *fixed legs.* Assume the swap is initiated at $T = 0$ and that the payments involved in the swap occur at intervals $T_1 = \delta$, $T_2 = 2\delta, \ldots$, $T_j = \delta j$, $\ldots$, $T_{n+1} = \delta(n + 1)$. This is called a swap of tenor T_{n+1}. The holder of a *receiver swap* receives fixed leg payments and pays floating legs. The holder of a *payer swap* receives floating legs and pays fixed legs.

Suppose the floating rate over interval $[T_{i-1}, T_i]$ is spot LIBOR $L(T_{i-1}, T_{i-1})$ (the tenor of LIBOR here is δ), and suppose also the fixed rate per annum is K. In a receiver swap on a principal of \$1, the holder pays $\delta L(T_{i-1}, T_{i-1})$ at T_i and receives δK. Problem 10.11 asks you to price this swap at time zero.

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2. *Swaps and swaptions*

a) Generally, the swap rate K of the contract when it is initiated is 0. This leads to formula (10.7.23) in Shreve. Show this formula is equivalent to

$$K = \frac{1 - B(0, T_{n+1})}{\delta \sum_{j=1}^{n+1} B(0, T_j)}.$$

b) The analysis of Exercise 10.11 in Shreve and the formula from part a) immediately generalize: The swap rate initiated at time T_m with payments at $T_{m+1}, \ldots, T_{n+1}$ is

$$K_m = \frac{1 - B(T_m, T_{n+1})}{\delta \sum_{j=m+1}^{n+1} B(T_m, T_j)}.$$

The denominator of this formula is the value at T_m of a portfolio consisting of δ units of $n + 1 - m$ zero coupon bonds, one for each of the maturity dates $T_{m+1}, \ldots, T_{n+1}$. The value of this portfolio at any time $t \leq T_m$ is

$$S_m^{n+1}(t) = \delta \sum_{j=m+1}^{n+1} B(t, T_j).$$

This is called the *accrual factor* or the *present value of a basis point.* Effectively, it is a discount factor for computing swap rates.

Now we want to generalize part a) of this problem. (Once you have understood the preliminary material, this is fairly simple.) Consider the swap that exchanges floating rate for fixed rate at times $T_{m+1}, \ldots, T_{n+1}$. This is called a $T_m \times (T_{n+1} - T_m)$ payer option. Show that, to the party receiving the fixed leg payments and paying the floating leg, the value of this swap for a fixed K at time $t \leq T_m$ is

$$\delta K \sum_{j=m+1}^{n+1} B(t, T_j) - \delta \sum_{j=m+1}^{n+1} B(t, T_j) L(t, T_{j-1})$$

From this, deduce that the *forward swap rate* at t, i.e. the swap rate that makes the value at t equal zero, is

$$R_m^{n+1}(t) = \frac{B(t, T_m) - B(t, T_{n+1})}{S_m^{n+1}(t)}.$$

3. *Swaps and Swaptions, continued.*

The accrual factor $S_m^{n+1}(t)$, defined in the previous problem for $0 \leq t \leq T_{m+1}$, can be used as a numéraire on this time interval. Let $\mathbf{Q}_m^{n+1}$ denote the risk-neutral (martingale) measure for numéraire $\{S_m^{n+1};\ 0 \leq t \leq T_{m+1}\}$; this is called the *annuity* or *swap measure.* Forward swap rates, as defined in problem 4, are martingales under

$\mathbf{Q_m^{n+1}}$ over the appropriate time interval, because the accrual rate appears in the denominator.

a) Let $\widetilde{\mathbf{P}}$ be the usual risk-neutral measure for prices denominated in dollars. Show

$$\frac{d\mathbf{Q}_m^{n+1}}{d\widetilde{\mathbf{P}}} = \frac{\sum_{j=m+1}^{n+1} D(T_{m+1})B(T_{m+1}, T_j)}{\sum_{j=m+1}^{n+1} B(0, T_j)}$$

b) The value of a payer swap at time T_m with fixed interest rate K is

$$V_m = \delta \sum_{j=m+1}^{n+1} B(T_m, T_j)L(T_m, T_{j-1}) - \delta K \sum_{j=m+1}^{n+1} B(T_m, T_j)$$

On the other hand we know that

$$0 = \delta \sum_{j=m+1}^{n+1} B(T_m, T_j)L(T_m, T_{j-1}) - \delta R_m^{n+1}(T_m) \sum_{j=m+1}^{n+1} B(T_m, T_j) \qquad (1)$$

where $R_m^{n+1}(t)$ denotes the forward swap rate as defined in problem 2.

A $T_m \times (T_{n+1} - T_m)$ payer swaption at strike K gives its holder the right, but not the obligation to enter into a $T_m \times (T_{n+1} - T_m)$ swap with fixed rate K at time T_m. Hence the payoff of this option is $\max\{V_m, 0\}$.

(i) First show that the payoff equals $S_m^{n+1}(T_m)(R_m^{n+1}(T_m) - K)^+$. Hint: V_m is not changed by subtracting the expression on the right-hand side of (1) from V_m.

(ii) Use this to show that the price at t of the swaption is

$$V(t) = S_m^{n+1}(t)E^{\mathbf{Q}_m^{n+1}}\left[(R_m^{n+1}(T_m) - K)^+ | \mathcal{F}(t)\right].$$

(iii) The final step is much like the derivation of Black's caplet formula (see Shreve). A *regular swap market model* assumes that there are deterministic volatility functions $\sigma_{0,n+1}(u), \ldots, \sigma_{n,n+1}(u)$ such that

$$dR_m^{n+1}(t) = R_m^{n+1}\sigma_{m,n+1}(t)\, dW_m^{n+1}(t),$$

where W_m^{n+1} is a Brownian motion under $\mathbf{Q}_m^{n+1}$. Assuming a regular swap model with given $\sigma_{m,n+1}(t)$ functions, find the price at time $t \le T_m$ of a $T_m \times (T_{n+1} - T_m)$ payer swaption. The answer is called Black's formula for swaptions.

For more background on swap market models and swaptions, see Björk, *Arbitrage Theory in Continuous Time*, Chapter 24.

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4. (See posted notes on Gaussian random vectors and processes for background on this problem.) Consider a multi-factor, Vasicek model of the form

$$dY(t) = GY(t)\,dt + B \cdot dW, \qquad R(t) = \delta_0 + \delta \cdot Y(t).$$

Here X is an m-vector valued process; G is an $m \times m$ matrix; B is an $m \times d$ matrix, and W is a d-dimensional Brownian motion; δ is an m-vector. Show that this is an affine-yield model of the type

$$B(t,T) = \exp\{-C(T-t) \cdot Y(t) - A(T-t)\},$$

where $C(T-t)$ is m-vector valued, and derive expressions for $C(T-t)$ and $A(T-t)$ in terms of B, δ_0, δ, and e^{sG}, $0 \le s \le T$.